Derivation Of The Butler Pinkerton Model

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The definition of risk in mathematical finance is the standard deviation of returns. The process of diversification involves placing risky assets that are not perfectly correlated into a portfolio such that a portion of the risk associated with the individual assets is eliminated. In other words, the risk of the whole is less than the sum of the parts, which is exactly what you want. Risk is composed of firm-specific risk (nonsystematic risk), which can be eliminated through diversification, and market risk (systematic risk), which cannot be eliminated through diversification.

Beta per the Capital Asset Pricing Model (CAPM) is the starting point when calculating a cost of capital. CAPM Beta assumes that the company being valued will be added to a well-diversified portfolio of assets. The diversified portfolio assumption is critical because it assumes that the portion of risk that is nonsystematic is eliminated once the asset is placed in the portfolio. The owners of small, non-public companies often have the value of the company as their only asset. In these cases the CAPM Beta does not apply. The Butler Pinkerton Model calculates a CAPM Beta adjusted for this lack of diversification. What follows is the derivation of the model assuming suboptimal diversification (no diversification or partial diversification).

Legend of Symbols

r_s	=	Expected annual return on an individual stock
r_m	=	Expected annual return on the market portfolio
r_f	=	Expected annual return on a riskless asset
σ_s	=	Standard deviation of individual stock annual returns
σ_m	=	Standard deviation of market portfolio annual returns
σ_p	=	Standard deviation of portfolio annual returns (portfolio of risky assets)
β_s	=	Individual stock beta coefficient
β_m	=	Market portfolio beta coefficient (equal to one by definition)
cov_{sm}	=	Covariance of individual stock returns and market portfolio returns
ρ_{sm}	=	Correlation of individual stock returns and market portfolio returns
ϕ	=	The market price of risk
λ	=	Percent of total risk that remains after diversification
w	=	Percent of wealth invested in private company stock

The Market Price of Risk

The risk-averse investor does not invest in a risky asset unless he or she expects a return in excess of the return from a riskless asset. Risk in this context is defined as the standard deviation of returns (volatility). The market price of risk is the additional return over the risk-free rate that risk-averse investors require per unit of volatility. The market price of risk is measured as the expected return in excess of the risk-free rate divided by the standard deviation of returns. The equation for the market price of risk is...

$$\phi = \frac{r_m - r_f}{\sigma_m} \tag{1}$$

Using the S&P 500 as a proxy for the market portfolio the average annual return and standard deviation of returns on this index over the period 1928 to 2008 was approximately 11% and 20%, respectively. Using the 10-year treasury note as a proxy for the riskless asset the annual return on this asset over this period was approximately 5%. The average market price of risk over the period 1928 to 2008 was...

$$\phi = \frac{0.11 - 0.05}{0.20} = 0.30 \tag{2}$$

Total risk (standard deviation of returns) is comprised of market (systematic) risk, which are risks that impact the market as a whole, and firm-specific (nonsystematic) risk, which are risks that are specific to the individual firm. We know from Modern Portfolio Theory that nonsystematic risk can be reduced or eliminated through diversification. We will define the variable λ as the percent of total risk that is not eliminated through diversification. The equation for the annual rate of return (cost of capital) required by risk-averse investors is...

$$r_s = r_f + \sigma_s \lambda \phi \tag{3}$$

Per equation (3) above the risk that is actually priced is $\sigma_s \lambda$. We know that this residual risk includes systematic risk as this cannot be diversified away. Residual risk may or may not include nonsystematic risk depending on how diversification is employed and if it is optimal.

The Efficient Frontier

Portfolios that have the highest expected return possible for any given amount of risk lie on the efficient frontier. A portfolio lies on the efficient frontier when all nonsystematic risk is removed from the portfolio through optimal diversification. Let's assume that portfolio p is a portfolio of risky assets such that...

N	=	Number of assets in the portfolio
w_i	=	Dollar weight of asset i in the portfolio
r_i	=	Expected return on asset i
σ_i	=	Standard deviation of asset i returns
cov_{ij}	=	Covariance of asset i returns with asset j returns
ρ_{ij}	=	Correlation of asset i returns with asset j returns

Expected portfolio return is a linear equation in that the expected portfolio return is a weighted average of the expected returns of individual assets in the portfolio. The equation for expected portfolio return (r_p) is...

$$r_p = \sum_{i=1}^{N} w_i r_i \tag{4}$$

Whereas the equation for portfolio return is linear the equation for portfolio variance is nonlinear and therefore is a candidate for optimization. The equation for portfolio variance (σ_p^2) is...

$$\begin{aligned} \sigma_p^2 &= \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j cov_{ij} \\ &= \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_i \sigma_j \rho_{ij} \end{aligned} \tag{5}$$

What happens to portfolio variance as N becomes large? As N becomes large no one asset dominates the portfolio. When this is the case we can make some simplifying assumptions. We will assume that all assets in the portfolio have equal weights such that w_i and w_j are each equal to $1/N$, and that all assets in the portfolio have equal volatilities such that σ_i and σ_j are each equal to σ_s. After making these substitutions equation (5) becomes...

$$\begin{aligned} \sigma_p^2 &= \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{N} \frac{1}{N} \sigma_s \sigma_s \rho_{ij} \\ &= \frac{\sigma_s^2}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \rho_{ij} \\ &= \frac{\sigma_s^2}{N^2} \left[\sum_{i=1}^{N} \sum_{j=1}^{N} \rho_{ij} | i \neq j + \sum_{i=1}^{N} \sum_{j=1}^{N} 1 | i = j \right] \end{aligned} \tag{6}$$

The pairwise correlation between the returns of asset i and asset j is the product of the correlation of asset i returns with market returns and asset j returns the market returns. The equation for the pairwise correlation of asset i and asset j returns where $i \neq j$ is...

$$\rho_{ij} = \rho_{im} \times \rho_{jm} \tag{7}$$

Just as we assumed that all assets in the portfolio have equal volatilities we will assume that all assets in the portfolio have equal correlations such what ρ_{im} and ρ_{jm} are each equal to ρ_{sm}. The equation for the pairwise correlation of asset i and asset j returns where $i \neq j$ becomes...

$$\rho_{ij} = \rho_{sm}^2 \tag{8}$$

If we substitute equation (8) into equation (6) the equation for portfolio variance becomes...

$$\begin{aligned}
\sigma_p^2 &= \frac{\sigma_s^2}{N^2}\bigg[N(N-1)\rho_{sm}^2 + N\bigg] \\
&= \frac{\sigma_s^2}{N}\bigg[(N-1)\rho_{sm}^2 + 1\bigg] \\
&= \sigma_s^2\bigg[\frac{(N-1)}{N}\rho_{sm}^2 + \frac{1}{N}\bigg] \qquad (9)
\end{aligned}$$

As N goes to infinity the equation for portfolio variance becomes...

$$\sigma_p^2 = \sigma_s^2\rho_{sm}^2 \tag{10}$$

The equation for portfolio return volatility, which is the square root of variance, becomes...

$$\sigma_p = \sigma_s\rho_{sm} \tag{11}$$

Total risk in equation (9) has two components, systematic risk and nonsystematic risk. The part of the equation that accumulates nonsystematic risk is $1/N$, which goes to zero as N goes to infinity. The part of the equation that accumulates systematic risk is $\rho_{sm}^2(N-1)/N$, which goes to ρ_{sm}^2 as N goes to infinity.

Key Point: As N goes to infinity all nonsystematic risk is removed from volatility such that portfolio residual risk equals $\sigma_s\rho_{sm}$.

Beta And The Capital Asset Pricing Model (CAPM)

Beta is a measure of a stock's volatility in relation to the market. The beta coefficient for an individual stock is a function of the covariance of individual stock returns with market portfolio returns and the variance of market portfolio returns. The equation for an individual stock's beta is...

$$\beta_s = \frac{cov(s,m)}{\sigma_m^2} \tag{12}$$

The Pearson correlation coefficient is a measure of the linear dependence of two variables. The equation for the correlation of individual stock returns with market portfolio returns is...

$$\rho_{sm} = \frac{cov(s,m)}{\sigma_s\sigma_m} \tag{13}$$

Since the beta equation (12) and the correlation equation (13) both have the covariance of individual stock returns with market portfolio returns in the numerator we can redefine beta to be a function of the correlation of individual stock returns with market portfolio returns and the product of their volatilities. The new equation for an individual stock's beta is...

$$\begin{aligned}
\beta_s\sigma_m^2 &= \rho_{sm}\sigma_s\sigma_m \\
\beta_s\sigma_m &= \rho_{sm}\sigma_s \\
\beta_s &= \frac{\sigma_s}{\sigma_m}\rho_{sm} \qquad (14)
\end{aligned}$$

The capital asset pricing model is used to determine a required rate of return on an asset. The equation for expected return (cost of capital) on an individual stock via the CAPM is...

$$r_s = r_f + \beta_s(r_m - r_f) \tag{15}$$

We can replace the β_s in equation (15) above with the definition of β as defined by equation (14). The equation for expected return becomes...

$$r_s = r_f + \frac{\sigma_s}{\sigma_m}\rho_{sm}(r_m - r_f) \tag{16}$$

We can replace the σ_m and $r_m - r_f$ in equation (16) above with the market price of risk as defined by equation (2). The equation for expected return becomes...

$$r_s = r_f + \sigma_s \rho_{sm} \phi \tag{17}$$

Note that the risk being priced in equation (17) above is equal to $\sigma_s \rho_{sm}$, which according to equation (11) is residual portfolio risk after all nonsystematic risk is removed.

Key Point: The CAPM beta assumes optimal diversification such that all nonsystematic risk is removed.

A Tale of Two Betas

Suboptimal diversification occurs when the investor chooses an asset mix that does not eliminate nonsystematic risk. Investors may choose to employ suboptimal diversification or suboptimal diversification may be the unavoidable result of market structure.

The investor who chooses suboptimal diversification in a complete market seeks a risk premium equal to total risk (systematic plus nonsystematic risk) times the market price of risk. The investor who chooses optimal diversification in a complete market seeks a risk premium equal to systematic risk times the market price of risk. The existence of competitors who are optimally diversified forces the investor who is suboptimally diversified to employ the CAPM equation, which prices systematic risk only, when pricing assets. Optimally diversified investors prevent asset prices from falling to the point where the expected return includes a premium for nonsystematic risk. Because both investors use the CAPM quation when pricing assets neither investor is compensated for nonsystematic risk. The pricing equation ***in a complete market***, which is derived from equation (15), is...

$$r_s = r_f + \beta_s(r_m - r_f) \tag{18}$$

The alternative is the investor who does not choose to be suboptimally diversified but rather is forced by market structure to be suboptimally diversified. An example is the small business owner who has all or a substantial portion of his or her net worth invested in a private business. In these cases optimal diversification is not an option. This investor seeks a risk premium equal to total risk (systematic plus nonsystematic risk) times the market price of risk. Because of market structure, lack of competitors and the fact that the business is not publicly traded means that there are no investors who employ optimal diversification so as to bid up asset prices such that investors are not compensated for nonsystematic risk. The pricing equation ***in an incomplete market***, which is derived from equation (3), is...

$$\begin{aligned} r_s &= r_f + \sigma_s \lambda \phi \\ &= r_f + \sigma_s \lambda \left[\frac{r_m - r_f}{\sigma_m}\right] \\ &= r_f + \frac{\sigma_s}{\sigma_m}\lambda \left[r_m - r_f\right] \\ &= r_f + \hat{\beta}_s(r_m - r_f) \end{aligned} \tag{19}$$

Where private company total beta is...

$$\hat{\beta}_s = \frac{\sigma_s}{\sigma_m}\lambda \tag{20}$$

Key Point: Incomplete markets allow for full pricing of nonsystematic risk.

Derivation of Private Company Beta

We will define the investor's portfolio as a portfolio of two assets, the stock of a private company and the market portfolio (S&P 500 Index). The percent of investor wealth allocated to the private company stock is w and the percent allocated to the market portfolio is $1 - w$. Total beta of the private company stock can be derived via equation (27) above. The only unknown in the equation is lambda (λ), which is the percent of total private company risk that is not diversified away via this asset allocation. What follows are the steps taken to solve for lambda.

The market portfolio by definition has zero nonsystematic risk so therefore we can use equation (17) to calculate its expected return. After substituting r_m for r_s, σ_m for σ_s and ρ_{mm} for ρ_{sm}, the equation for expected market portfolio return becomes...

$$r_m = r_f + \sigma_m \phi \tag{21}$$

Because the market portfolio has zero nonsystematic risk all benefits from diversification (combining the market portfolio with the private company stock) will be used to reduce the private company stock stand-alone beta. Even after diversification, which in this case is suboptimal, the private company stock will still have nonsystematic risk albeit less than if it were a stand-alone investment. We want to price for this residual risk so we will use equation (3) to to calculate its expected return. The equation for expected return on the private company stock is...

$$r_s = r_f + \sigma_s \lambda \phi \tag{22}$$

The expected return on the investor's portfolio is a weighted average of the portfolio's individual asset expected returns. The equation for expected portfolio return, which is derived from equation (4), is...

$$\begin{aligned} r_p &= w r_s + (1-w) r_m \\ &= w[r_f + \sigma_s \lambda \phi] + (1-w)[r_f + \sigma_m \phi] \\ &= r_f + w \sigma_s \lambda \phi + (1-w) \sigma_m \phi \end{aligned} \tag{23}$$

The equation for portfolio return variance, which is derived from equation (5), is...

$$\sigma_p^2 = w^2 \sigma_s^2 + (1-w)^2 \sigma_m^2 + 2w(1-w) \sigma_s \sigma_m \rho_{sm} \tag{24}$$

Because we want to price for all risk in the portfolio (systematic and nonsystematic) the equation for portfolio expected return as a function of portfolio variance and the market price of risk is...

$$r_p = r_f + \sigma_p \phi \tag{25}$$

We can now set equations (23) and (25) equal to each other and solve for lambda. The percent of private company total risk that is not diversified away (lambda) is...

$$\begin{aligned} r_f + w \sigma_s \lambda \phi + (1-w) \sigma_m \phi &= r_f + \sigma_p \phi \\ w \sigma_s \lambda \phi + (1-w) \sigma_m \phi &= \sigma_p \phi \\ w \sigma_s \lambda + (1-w) \sigma_m &= \sigma_p \\ \lambda &= \frac{\sigma_p - (1-w)\sigma_m}{w \sigma_s} \end{aligned} \tag{26}$$

Total beta (CAPM Beta adjusted for the lack of diversification) per equation (27) is therefore...

$$\begin{aligned} \hat{\beta}_s &= \frac{\sigma_s}{\sigma_m} \lambda \\ &= \frac{\sigma_s}{\sigma_m} \left[\frac{\sigma_p - (1-w)\sigma_m}{w \sigma_s} \right] \\ &= \frac{\sigma_p - (1-w)\sigma_m}{w \sigma_m} \end{aligned} \tag{27}$$

A Hypothetical Case

We are asked to calculate the total beta of a private company assuming the following parameters...

Beta coefficient of proxy stock (s)	=	2.00
Volatility of market returns	=	0.20
Correlation of proxy stock returns with market returns	=	0.50

Using the equations above we produce the following table of private company betas assuming different portfolio asset allocations.

Private company weight	Private company beta	Lambda	Notes
100%	4.00	1.0000	
90%	3.95	0.9864	
80%	3.88	0.9702	
70%	3.80	0.9595	See example calculation below
60%	3.70	0.9262	
50%	3.58	0.8956	
40%	3.42	0.8561	
30%	3.21	0.8036	
20%	2.93	0.7321	
10%	2.53	0.6331	
1%	2.06	0.5149	

Example calculations using a 70% private company and 30% market portfolio asset allocation:

Step 1: Calculate volatility of proxy stock returns via equation (14):

$$\begin{aligned}\beta_s &= \frac{\sigma s}{\sigma_m}\rho_{sm} \\ \sigma_s &= \frac{\beta_s \sigma_m}{\rho_{sm}} \\ \sigma_s &= \frac{2.00 \times 0.20}{0.50} \\ \sigma_s &= 0.80\end{aligned} \tag{28}$$

Step 2: Calculate portfolio return volatility via equation (24):

$$\begin{aligned}\sigma_p^2 &= w^2\sigma_m^2 + (1-w)^2\sigma_s^2 + 2w(1-w)\sigma_m\sigma_s\rho_{sm} \\ \sigma_p^2 &= (0.30^2)(0.20^2) + (0.70^2)(0.80^2) + (2)(0.30)(0.70)(0.20)(0.80)(0.50) \\ \sigma_p^2 &= 0.3508 \\ \sigma_p &= 0.5923\end{aligned} \tag{29}$$

Step 3: Calculate lambda via equation (26):

$$\begin{aligned}\lambda &= \frac{\sigma_p - w\sigma_m}{(1-w)\sigma_s} \\ &= \frac{0.5923 - (0.30)(0.20)}{(0.70)(0.80)} \\ &= 0.9505\end{aligned} \tag{30}$$

Step 4: Calculate private company total beta via equation (27):

$$\begin{aligned}\hat{\beta}_s &= \frac{\sigma_s}{\sigma_m}\lambda \\ &= \frac{0.80}{0.20}0.9505 \\ &= 3.80\end{aligned} \tag{31}$$